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Twisted Hopf Algebras, Ringel–Hall Algebras, and Green's Categories[1]

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The concept and some basic properties of a twisted Hopf algebra are introduced and investigated. Its unique difference from a Hopf algebra is that the comultiplication $\delta: A \to A \otimes A$ is an algebra homomorphism, not for the componentwise multiplication on $A \otimes A$, but for the twisted multiplication on $A \otimes A$ given by Lusztig's rule.

Also, it is proved that any object A in Green's category has a twisted Hopf algebra structure, any morphism between objects is a twisted Hopf algebra homomorphism, the antipode s of A is self-adjoint under the Lusztig form $(-, -)$ on A, and the Green polynomials $M_{a,b}(t)$ share a so-called cyclic-symmetry.

As examples, the twisted Ringel–Hall algebras, Ringel's twisted composition algebras, Lusztig's free algebras $'\mathbf{F}$ and non-degenerate algebras $\mathbf{F}$, the positive part U^+ of the Drinfeld–Jimbo quantized enveloping algebras U, and Rosso's quantum shuffle algebra $T(V)$ all are twisted Hopf algebras. The antipode and its inverse for a twisted Ringel–Hall are explicitly given, and all δ-primitive elements are determined. © 2000 Academic Press

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1. INTRODUCTION

Some basic properties for Ringel's composition algebras (see [R3]) and Lusztig's algebras $'\mathbf{F}$ and $\mathbf{F}$ associated with any datum $(I, \cdot)$ over any field K (see [L]) have been summarized by Green as axioms in the category $\mathscr{L}(K, c, I, \cdot)$ (see [G1, G2]): roughly speaking, an object A belonging to $\mathscr{L}(K, c, I, \cdot)$ has an $\mathbb{N}_0 I$-graded K-algebra structure with generators indexed by I; a coassociative comultiplication δ such that the generators are δ-primitive elements and that $\delta: A \to A \otimes A$ is an algebra homomorphism, where the algebra structure on $A \otimes A$ is given by Lusztig's rule; and a symmetric bilinear form $(-,-)$ on A with some special properties. Some interesting and important properties for all algebras in $\mathscr{L}(K, c, I, \cdot)$ have been derived in [G1, G2], for example, the existence of the polynomials $M_{a,b}(t)$! With these properties, Green has proved the famous algebra isomorphism $\mathscr{C}(\Delta, d) \cong U^+$ (see [G1, G2, R3, R4, R5, R6]) via Lusztig's isomorphism $\mathbf{f} \cong U^+$ (see [L]), where $\mathscr{C}(\Delta, d)$ and U^+ are, respectively, the twisted generic composition algebra and the positive part of the Drinfeld–Jimbo quantized enveloping algebra U, both of type (Δ, d), where Δ is an arbitrary symmetrizable generalized Cartan matrix with symmetrization d, in Kac's sense [K].

Of course, one can define a counit ε of $A \in \mathscr{L}(K, c, I, \cdot)$ to be the projection onto $A_0 = K$. In this way, A becomes a χ-bialgebra in Ringel's sense [R6]. However, it seems to be of interest that A has an antipode s compatible with the bilinear form $(-,-)$ on A. This is one of the main motivations of this paper.

On the other hand, since a strong feature of a quantum group is its Hopf algebra structure, and since all existing Hopf operations are not closed inside U^+ (see, e.g., [D, Jan, Jim, Jos, L]), we naturally hope that inside U^+ there is a "nearly" Hopf algebra structure. A twisted Hopf algebra structure as defined in 2.8 seems to be a natural candidate. The unique difference between a twisted Hopf algebra and Hopf algebra is that the comultiplication $\delta: A \to A \otimes A$ is an algebra homomorphism, not for the componentwise multiplication on $A \otimes A$, but for the twisted multiplication on $A \otimes A$ given by Lusztig's rule. This idea of a twisted multiplication should not be considered a disadvantage. On the contrary, it can be used for any algebra and any coalgebra, as Lusztig [L] and Ringel [R4, R6] did, and a twisted Hopf algebra shares some basic properties of a Hopf algebra, which is worthy to be studied.

For any field K, a non-zero element c in K, and any datum $(I, \chi = \cdot)$, the concept of a χ-Hopf algebra is introduced in Section 2. It turns out that a χ-Hopf algebra is exactly a χ-bialgebra in Ringel's sense and that the antipode s of a χ-Hopf algebra A gives a graded algebra anti-isomorphism $s: A \to A_{\chi^T}$ and a graded coalgebra anti-isomorphism $s: A_\chi \to A$, where the algebra and coalgebra structure on A_χ are given by twisted rules; see Theorem 2.10. Moreover, any χ-Hopf algebra can be embedded in a Hopf algebra; see Theorem A in the Appendix by the referee.

Now, any object A in Green's category $\mathscr{L}(K, c, I, \cdot)$ has a χ-Hopf algebra structure. The antipode s of A turns out to be self-adjoint under the bilinear form $(-,-)$ on A, and a morphism between two objects turns out to be a χ-Hopf algebra homomorphism, and all δ-primitive elements are determined; see Theorem 4.4. By this self-adjointness of s under the bilinear form $(-,-)$ on A, it is proved that the Green polynomials $M_{a,b}(t)$ share a so-called cyclic-symmetry; see Theorem 4.7.

Since a χ-Hopf algebra is just a χ-bialgebra, it follows that there are a lot of examples available: the twisted Ringel–Hall algebras, Ringel's twisted composition algebras, Lusztig's free algebras $'\mathbf{F}$ and non-degenerate algebras $\mathbf{F}$ associated with any datum over any field K, and the positive part U^+ of the Drinfeld–Jimbo quantized enveloping algebras U.

Given any symmetrizable generalized Cartan matrix Δ with symmetrization d or, equivalently, a Cartan datum $(I, \cdot)$ in Lusztig's sense [L], there is a finite-dimensional hereditary algebra Λ of type $(I, \cdot)$, over any finite field k, with $v_k = |k|^{1/2}$. Consider the twisted Ringel–Hall algebra $\mathscr{H}(\Lambda)$ and Ringel's (twisted) composition algebra $\mathscr{C}(\Lambda)$ as χ-Hopf algebras; here $\chi = \cdot$ is just the symmetrization of the Ringel form. The explicit expressions of the antipode s and its inverse of $\mathscr{H}(\Lambda)$ and $\mathscr{C}(\Lambda)$ can be easily derived, by using Theorem A in the Appendix given by the referee and the antipode of the "extended" (twisted) Ringel–Hall algebra given by Xiao [X]; see Lemma 3.3. Then it is proved that s of $\mathscr{H}(\Lambda)$ is self-adjoint under the bilinear form $(-,-)$ on $\mathscr{H}(\Lambda)$; see Theorem 3.5.

By using an observation of Sevenhant and Van den Bergh, all δ-primitive elements of $\mathscr{H}(\Lambda)$ are determined, and then a sufficient and necessary condition for $\mathscr{C}(\Lambda) = \mathscr{H}(\Lambda)$ is given; see Theorem 3.7.

Finally, we include some direct calculations in $\mathscr{H}(\Lambda)$ in Section 5.

2. TWISTED COALGEBRAS AND χ-HOPF ALGEBRAS

2.1. Throughout this section, let K be a field, let c be a non-zero element in K, and let I be a set. Write $\otimes$ for $\otimes_K$. Denote by $\mathbb{Z}I$ the free abelian group with I as basis. An element in $\mathbb{Z}I$ is written as $x = (x_i)_{i \in I}$ with $x_i \in \mathbb{Z}$, where $x_i = 0$ for almost all $i \in I$. Denote by $\mathbb{N}_0 I$ the subset $\{x = (x_i)_{i \in I} \in \mathbb{Z}I \mid x_i \in \mathbb{N}_0\}$.

An $\mathbb{N}_0 I$-graded algebra A means an associative K-algebra with a direct decomposition of K-spaces $A = \oplus_{x \in \mathbb{N}_0 I} A_x$ with $A_0 = K$, such that $A_x A_y \subseteq A_{x+y}$.

In the following, if $a \in A_x$, then x is called the degree of a and is denoted by $|a| = x$.

By definition [R6, p. 206], an $\mathbb{N}_0 I$-graded K-coalgebra $A = (A, \delta, \varepsilon)$ is a graded K-space $A = \oplus_{x \in \mathbb{N}_0 I} A_x$ with $A_0 = K$, and with K-linear maps $\delta\colon A \to A \otimes A$ and $\varepsilon\colon A \to K$ satisfying the following conditions:

(i) δ is a coassociative comultiplication, i.e., $(\mathrm{id} \otimes \delta)\delta = (\delta \otimes \mathrm{id})\delta$;

(ii) ε is the projection onto $A_0 = K$, i.e., $\varepsilon(A_x) = 0$ for $x \neq 0$ and $\varepsilon(1) = 1$;

(iii) ε is a counit, i.e., $(\mathrm{id} \otimes \varepsilon)\delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id})\delta$;

(iv) δ respects the grading, i.e., $\delta(A_z) \subseteq \oplus_{x+y=z} A_x \otimes A_y$.

LEMMA 2.2. *Let A be an $\mathbb{N}_0 I$-graded K-coalgebra. Then*

(i) $\delta(1) = 1 \otimes 1$.

(ii) *For $a \in A_d$ with $d \neq 0$ we have*

$$\delta(a) = a \otimes 1 + 1 \otimes a + \sum_{x+y=d;\, x, y \neq 0} a_x \otimes a'_y,$$

where $a_x \in A_x$, $a'_y \in A_y$.

In particular, we have $\delta(a) = a \otimes 1 + 1 \otimes a$ for $a \in A_i$, $i \in I$.

Proof. Since δ respects the grading, it follows that $\delta(1) = 1 \otimes a'$ with $a' \in A_0 = K$. Then $1 = (\mathrm{id} \otimes \varepsilon)\delta(1) = (\mathrm{id} \otimes \varepsilon)(1 \otimes a') = a'$. For $a \in A_d$ with $d \neq 0$, since δ respects the grading, it follows that we can write

$$\delta(a) = 1 \otimes a' + a'' \otimes 1 + \sum_{x+y=d;\, x, y \neq 0} a_x \otimes a'_y,$$

where $a', a'' \in A_d$, $a_x \in A_x$, and $a'_y \in A_y$. Since $\varepsilon(A_x) = 0$ for $x \neq 0$, it follows that $a = (\mathrm{id} \otimes \varepsilon)\delta(a) = a''$ and $a = (\varepsilon \otimes \mathrm{id})\delta(a) = a'$. ∎

2.3. Let $\chi\colon \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ be a (not necessarily symmetric) bilinear form. Define a new bilinear form $\chi^T\colon \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ by

$$\chi^T(x, y) = \chi(y, x).$$

Let A be an $\mathbb{N}_0 I$-graded algebra. Ringel [R4] has introduced a new multiplication $*$ on A: for $a \in A_x$, $b \in A_y$, define

$$a * b = c^{\chi(|a|, |b|)} ab. \tag{2.1}$$

Then there is a unique $\mathbb{N}_0 I$-graded, associative K-algebra structure on A with multiplication $*$. Following Ringel, denote this new algebra by A_χ, and denote the new multiplication map by m_χ, if the original multiplication map of A is denoted by m.

Dually, let $A = (A, \delta, \varepsilon)$ be an $\mathbb{N}_0 I$-graded coalgebra. Consider a new K-linear comultiplication δ_χ on A defined by

$$\delta_\chi(a) = \sum c^{\chi(|a_1|, |a_2|)}(a_1 \otimes a_2) \tag{2.2}$$

for all $a \in A$, where $\delta(a) = \Sigma a_1 \otimes a_2$ is Sweedler's notation with all factors a_1, a_2 homogeneous. Note that this construction of δ_χ has been introduced by Lusztig for $'\mathbf{f}$ and $\mathbf{f}$ in [L, p. 6].

LEMMA 2.4. *Let A be an $\mathbb{N}_0 I$-graded coalgebra. Then $(A, \delta_\chi, \varepsilon)$ is again an $\mathbb{N}_0 I$-graded coalgebra, which is denoted by A_χ.*

Proof. Note that δ_χ respects the grading, since δ does so. Since ε is the projection onto K and ε is the counit of A, it is easy to see that ε is the counit of A_χ.

It remains to check the coassociativity of δ_χ. For $a \in A$, by the coassociativity of δ we can write

$$(\mathrm{id} \otimes \delta)\delta(a) = (\delta \otimes \mathrm{id})\delta(a) = \sum a_1 \otimes a_2 \otimes a_3,$$

with all factors being homogeneous. It follows that we can write

$$(\mathrm{id} \otimes \delta_\chi)\delta_\chi(a) = \sum c_{a_1, a_2, a_3} a_1 \otimes a_2 \otimes a_3,$$
$$(\delta_\chi \otimes \mathrm{id})\delta_\chi(a) = \sum c'_{a_1, a_2, a_3} a_1 \otimes a_2 \otimes a_3,$$

where

$$\begin{aligned} c_{a_1, a_2, a_3} &= c^{\chi(|a_1|, |a_2| + |a_3|)} c^{\chi(|a_2|, |a_3|)} \\ &= c^{\chi(|a_1|, |a_2|) + \chi(|a_1|, |a_3|) + \chi(|a_2|, |a_3|)} \\ &= c^{\chi(|a_1| + |a_2|, |a_3|)} c^{\chi(|a_1|, |a_2|)} \\ &= c'_{a_1, a_2, a_3}, \end{aligned}$$

where we have used that $\delta_\chi(A_z) \subseteq \oplus_{x+y=z} A_x \otimes A_y$. This completes the proof. ∎

LEMMA 2.5. *Let $A = (A, \delta, \varepsilon)$ be an $\mathbb{N}_0 I$-graded K-coalgebra, and let χ: $\mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ be a bilinear form. Then ${}_\chi A = (A, {}_\chi\delta, \varepsilon)$ is again an $\mathbb{N}_0 I$-graded K-coalgebra, where the K-linear map ${}_\chi\delta$: $A \to A \otimes A$ is defined by*

$${}_\chi\delta(a) = \sum c^{-\chi(|a_1|, |a_2|)}(a_2 \otimes a_1) \tag{2.3}$$

for $a \in A$, where $\delta(a) = \Sigma a_1 \otimes a_2$ with all factors homogeneous.

Proof. Note that ${}_\chi\delta$ is exactly $T\delta_{-\chi}$, where T is the twisted map [S, p. 49]

$$T: A \otimes A \to A \otimes A, \qquad a \otimes b \mapsto b \otimes a.$$

Thus, the assertion follows from Lemma 2.4 and the fact that if (A, δ, ε) is an $\mathbb{N}_0 I$-graded coalgebra, then so is $(A, T\delta, \varepsilon)$. ∎

2.6. For a bilinear form $\chi\colon \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$, consider the corresponding bilinear form $(\mathbb{Z}I)^2 \times (\mathbb{Z}I)^2 \to \mathbb{Z}$, again denoted by χ, which is defined by

$$\chi(x_1, x_2, y_1, y_2) = \chi(x_2, y_1) \tag{2.4}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{Z}I$.

Let A be an $\mathbb{N}_0 I$-graded algebra. Then $A \otimes A$ is an $(\mathbb{N}_0 I)^2$-graded algebra with componentwise multiplication, where $(A \otimes A)_{(x, y)} = A_x \otimes A_y$ for $x, y \in \mathbb{N}_0 I$. Note that $(\mathbb{Z}I)^2 = \mathbb{Z}I'$, and $(\mathbb{N}_0 I)^2 = \mathbb{N}_0 I'$, where I' is a set with $|I'| = 2|I|$. Thus, by applying 2.3 to the tensor algebra $A \otimes A$, we obtain the algebra $(A \otimes A)_\chi$. Thus, $(A \otimes A)_\chi$ is an $(\mathbb{N}_0 I)^2$-graded, associative algebra with multiplication $*$ given by Lusztig's rule [L, p. 3],

$$(a_1 \otimes a_2) * (b_1 \otimes b_2) = c^{\chi(|a_2|, |b_1|)}(a_1 b_1 \otimes a_2 b_2) \tag{2.5}$$

for homogeneous elements $a_1, a_2, b_1, b_2 \in A$. Note that here we do not assume that χ is symmetric.

Dually, let $A = (A, \delta, \varepsilon)$ be an $\mathbb{N}_0 I$-graded coalgebra. Then $A \otimes A$ is an $(\mathbb{N}_0 I)^2$-graded coalgebra with comultiplication $(\mathrm{id} \otimes T \otimes \mathrm{id})(\delta \otimes \delta)$, again denoted by $\delta \otimes \delta$ if no confusion is caused, where T is the twisted map. Thus, if $\delta(a) = \sum a_1 \otimes a_2$ and $\delta(b) = \sum b_1 \otimes b_2$, then

$$(\delta \otimes \delta)(a \otimes b) = \sum a_1 \otimes b_1 \otimes a_2 \otimes b_2.$$

Now applying Lemma 2.4 we obtain the $(\mathbb{N}_0 I)^2$-graded coalgebra $(A \otimes A)_\chi = (A \otimes A, (\delta \otimes \delta)_\chi, \varepsilon \otimes \varepsilon)$ with comultiplication

$$(\delta \otimes \delta)_\chi(a \otimes b) = \sum c^{\chi(|b_1|, |a_2|)}(a_1 \otimes b_1 \otimes a_2 \otimes b_2), \tag{2.6}$$

where $\delta(a) = \sum a_1 \otimes a_2$ and $\delta(b) = \sum b_1 \otimes b_2$ such that all a_1, a_2, b_1, b_2 are homogeneous.

LEMMA 2.7. *Let χ be a bilinear form: $\mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$. Let $A = (A, m, e)$ be an $\mathbb{N}_0 I$-graded algebra and let $A = (A, \delta, \varepsilon)$ be an $\mathbb{N}_0 I$-graded coalgebra. Then*

(i) *$\varepsilon\colon A \to K$ is an algebra homomorphism.*

(ii) *$e\colon K \to A$ is a coalgebra homomorphism.*

(iii) *$\delta\colon A \to (A \otimes A)_\chi$ is an algebra homomorphism if and only if $m\colon (A \otimes A)_{\chi^T} \to A$ is a coalgebra homomorphism, where the algebra on $(A \otimes A)_\chi$ and the coalgebra structure on $(A \otimes A)_{\chi^T}$ are defined by* (2.5) *and* (2.6), *respectively.*

Proof. (i) is clear, since by definition ε is the projection onto $K = A_0$. By Lemma 2.2 we have $\delta(1) = 1 \otimes 1$; from this (ii) follows.

Since $\varepsilon(A_x) = 0$ for $x \neq 0$ and $\varepsilon(1) = 1$, it follows that $\varepsilon m = \varepsilon \otimes \varepsilon$. For any $a, b \in A$, let $\delta(a) = \Sigma a_1 \otimes a_2$, $\delta(b) = \Sigma b_1 \otimes b_2$, with all factors homogeneous. Now, δ: $A \to (A \otimes A)_\chi$ is an algebra homomorphism if and only if $\delta(ab) = \delta(a) * \delta(b)$, and if and only if

$$\begin{aligned}\delta(ab) &= \sum c^{\chi(|a_2|, |b_1|)} a_1 b_1 \otimes a_2 b_2 \\ &= \sum c^{\chi^T(|b_1|, |a_2|)} a_1 b_1 \otimes a_2 b_2 \\ &= (m \otimes m)(\delta \otimes \delta)_{\chi^T}(a \otimes b),\end{aligned}$$

which is exactly the claim that m: $(A \otimes A)_{\chi^T} \to A$ is a coalgebra homomorphism. ∎

Now we introduce the notation of a twisted Hopf algebra.

DEFINITION 2.8 [R6, p. 207]. Let K, c, I be as in 2.1 and let χ be an arbitrary bilinear form: $\mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$.

(1) A K-module A is called a (K, c, I, χ)-bialgebra, or simply, a χ-bialgebra, provided that the following two conditions are satisfied:

R1. $A = \oplus_{x \in \mathbb{N}_0 I} A_x$ is an $\mathbb{N}_0 I$-graded, both K-algebra and K-coalgebra in the sense of 2.1.

Denote the multiplication map of A by m, the unit map by e, the comultiplication of A by δ, and the counit by ε.

R2. δ: $A \to (A \otimes A)_\chi$ is an algebra homomorphism

(2) A (K, c, I, χ)-bialgebra $A = (A, m, e, \delta, \varepsilon)$ is called a (K, c, I, χ)-Hopf algebra or, simply, a χ-Hopf algebra, provided that the following condition is satisfied:

R3. There is a K-linear map s: $A \to A$ such that

$$m(\mathrm{id} \otimes s)\delta = e\varepsilon = m(s \otimes \mathrm{id})\delta. \tag{2.7}$$

The map s is called an antipode of A.

2.9. The notion of a χ-bialgebra has been introduced by Ringel in [R6], in a more general situation than 2.8. Notice that saying $A = (A, m, e, \delta, \varepsilon)$ is a (K, c, I, χ)-bialgebra is equivalent to saying it is a $(K, c^{-1}, I, -\chi)$-bialgebra.

Let $A = (A, m, e)$, and let $C = (C, \delta, \varepsilon)$ be a K-algebra and a K-coalgebra, respectively. Recall that the convolution product $\star$ in $\mathrm{Hom}_K(C, A)$ is defined by

$$(f \star g)(x) = m(f \otimes g)\delta(x)$$

for $f, g \in \mathrm{Hom}_K(C, A)$, $x \in C$. Then $\mathrm{Hom}_K(C, A)$ becomes an associative K-algebra with unit $e\varepsilon$, with the multiplication $\star$; see [S, p. 69].

Thus, for a χ-bialgebra $(A, m, e, \delta, \varepsilon)$, the condition **R3** is equivalent to the following:

R3′. There is a K-linear map s: $A \to A$ such that in the convolution algebra $\mathrm{Hom}_K(A, A)$ there holds

$$s \star \mathrm{id} = \mathrm{id} \star s = e\varepsilon. \tag{2.8}$$

Thus, a χ-Hopf algebra has a unique antipode.

The main result of this section is as follows.

THEOREM 2.10. *Let $A = (A, m, e, \delta, \varepsilon)$ be a (K, c, I, χ)-bialgebra. Then we have*

(i) *There is a unique χ-Hopf algebra structure on A, with antipode denoted by s.*

(ii) *s: $A \to A$ is an $\mathbb{N}_0 I$-graded map.*

(iii) *s: $A \to A_{\chi^T}$ is an algebra anti-homomorphism.*

(iv) *s: $A_\chi \to A$ is a coalgebra anti-homomorphism.*

(v) *${}_{\chi^T}A = (A, m, e, {}_{\chi^T}\delta, \varepsilon)$ is a $(K, c, I, -\chi^T)$-Hopf algebra with antipode s^{-1}. In particular, s is invertible.*

Proof. (i) This has been proved by Zelevinsky in different terminology; see [Z, p. 152, Proposition A1.6]. We include here a more direct proof.

Let $A = \oplus_{x \in \mathbb{N}_0 I} A_x$. Define a K-linear map s_r: $A \to A$ inductively: define $s_r(1) = 1$; for $a \in A_d$ with $d \neq 0$, by Lemma 2.2 we have

$$\delta(a) = a \otimes 1 + 1 \otimes a + \sum_{x+y=d;\, x, y \neq 0} a_x \otimes a'_y,$$

where $a_x \in A_x$, $a'_y \in A_y$. Since $x, y < d$ (the partial order of $\mathbb{N}_0 I$ used here is componentwise), it follows from induction that $s_r(a'_y)$ has been well defined, and then define

$$s_r(a) = -a - \sum_{x+y=d;\, x, y \neq 0} a_x s_r(a'_y).$$

In this way, we have $m(\mathrm{id} \otimes s_r)\delta = e\varepsilon$, which means that s_r is a right inverse of id in the algebra $\mathrm{Hom}_K(A, A)$ with multiplication given by convolution $\bigstar$. Similarly, one gets a left inverse s_l of id. It follows that $s_r = s_l = s$ is the inverse of id; i.e., s is the antipode of A.

(ii) By the inductive construction of s in (i), one sees that s is an $\mathbb{N}_0 I$-graded map.

(iii) Since $s(1) = 1$, it remains to prove $s(ab) = s(b) * s(a)$ for $a, b \in A$ or, equivalently, $sm = m_{\chi^T}(s \otimes s)T$, where T is the twisted map. Since $e(\varepsilon \otimes \varepsilon)$ is the unit of the convolution algebra $\mathrm{Hom}_K((A \otimes A)_{\chi^T}, A)$, it follows that it suffices to prove that there holds $(sm) \bigstar m = m \bigstar (m_{\chi^T}(s \otimes s)T) = e(\varepsilon \otimes \varepsilon)$ in the convolution algebra $\mathrm{Hom}_K((A \otimes A)_{\chi^T}, A)$.

Let $a, b \in A$ be homogeneous, $\delta(a) = \sum a_1 \otimes a_2$, and $\delta(b) = \sum b_1 \otimes b_2$ with all factors homogeneous. Then by **R2** and (2.5) we have $\delta(ab) = \sum c^{\chi(|a_2|, |b_1|)}(a_1 b_1 \otimes a_2 b_2)$, and by (2.6) there holds the following:

$$\begin{aligned}((sm) \bigstar m)(a \otimes b) &= m(sm \otimes m)(\delta \otimes \delta)_{\chi^T}(a \otimes b)\\ &= \sum c^{\chi^T(|b_1|, |a_2|)} s(a_1 b_1) a_2 b_2\\ &= m(s \otimes \mathrm{id})\delta(ab)\\ &= \varepsilon(ab) = (e(\varepsilon \otimes \varepsilon))(a \otimes b).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\big(m \bigstar \big(m_{\chi^T}(s \otimes s)T\big)\big)(a \otimes b)\\ &\quad = m\big(m \otimes m_{\chi^T}(s \otimes s)T\big)(\delta \otimes \delta)_{\chi^T}(a \otimes b)\\ &\quad = \sum c^{\chi^T(|b_1|, |a_2|)} m\big(m \otimes m_{\chi^T}(s \otimes s)T\big)(a_1 \otimes b_1 \otimes a_2 \otimes b_2)\\ &\quad = \sum c^{\chi(|a_2|, |b_1|)} a_1 b_1 m_{\chi^T}(s(b_2) \otimes s(a_2))\\ &\quad = \sum c^{\chi(|a_2|, |b_1|) + \chi^T(|b_2|, |a_2|)} a_1 b_1 s(b_2) s(a_2)\\ &\quad = \sum c^{\chi(|a_2|, |b|)} a_1 b_1 s(b_2) s(a_2)\\ &\quad = \sum c^{\chi(|a_2|, |b|)} a_1 s(a_2) \varepsilon(b).\end{aligned}$$

It follows that if $b \notin A_0$, then $\varepsilon(b) = 0$, and hence

$$\big(m \bigstar \big(m_{\chi^T}(s \otimes s)T\big)\big)(a \otimes b) = 0 = (e(\varepsilon \otimes \varepsilon))(a \otimes b),$$

and if $b \in A_0$, then $\chi(|a_2|, |b|) = 0$, and hence

$$\left(m \star \left(m_{\chi^T}(s \otimes s)T\right)\right)(a \otimes b) = \sum a_1 s(a_2)\varepsilon(b) = m(\mathrm{id} \otimes s)\delta(a)\varepsilon(b)$$
$$= \varepsilon(a)\varepsilon(b) = (e(\varepsilon \otimes \varepsilon))(a \otimes b).$$

This proves (iii).

(iv) Since $s(A_x) \subseteq A_x$, $s(1) = 1$, $\varepsilon(A_x) = 0$ for $x \neq 0$, and $\varepsilon(1) = 1$, it follows from (ii) that $\varepsilon s = \varepsilon$.

It remains to prove that $\delta s = T(s \otimes s)\delta_\chi$. Note that $(e \otimes e)\varepsilon$ is the map which takes $a \in A$ to $\varepsilon(a)(1 \otimes 1) = \varepsilon(a) \otimes 1 = 1 \otimes \varepsilon(a)$. Since $(e \otimes e)\varepsilon$ is the unit of the convolution algebra $\mathrm{Hom}_K(A, (A \otimes A)_\chi)$, it follows that it suffices to prove $(\delta s) \star \delta = \delta \star (T(s \otimes s)\delta_\chi) = (e \otimes e)\varepsilon$ in the convolution algebra $\mathrm{Hom}_K(A, (A \otimes A)_\chi)$. Let a be a homogeneous element in A. Since δ is an algebra homomorphism from A to $(A \otimes A)_\chi$, it follows that

$$\begin{aligned}((\delta s) \star \delta)(a) &= \sum \delta s(a_1) * \delta(a_2) \\ &= \sum \delta(s(a_1)a_2) = \delta m(s \otimes \mathrm{id})\delta(a) \\ &= \delta\varepsilon(a) = \varepsilon(a)(1 \otimes 1) \\ &= (e \otimes e)\varepsilon(a).\end{aligned}$$

Let $\delta(a) = \Sigma a_1 \otimes_2$, $\delta(a_1) = \Sigma a_{11} \otimes a_{12}$, and $\delta(a_2) = \Sigma a_{21} \otimes a_{22}$ with all factors homogeneous. Then

$$|a_{11}| + |a_{12}| = |a_1|, \qquad |a_{21}| + |a_{22}| = |a_2|, \qquad |a_1| + |a_2| = |a|,$$

and we have

$$\left(\delta \star \left(T(s \otimes s)\delta_\chi\right)\right)(a) = \sum c^{\chi(|a_{21}|, |a_{22}|)}(a_{11} \otimes a_{12}) * (s(a_{22}) \otimes s(a_{21}))$$
$$= \sum c^{\chi(|a_{21}|, |a_{22}|) + \chi(|a_{12}|, |a_{22}|)} a_{11}s(a_{22}) \otimes a_{12}s(a_{21}).$$

On the other hand, by using the coassociativity of δ, we have

$$(\delta \otimes \delta)\delta = (\mathrm{id} \otimes \delta \otimes \mathrm{id})(\mathrm{id} \otimes \delta)\delta,$$

and hence we have

$$\sum a_{11} \otimes a_{12} \otimes a_{21} \otimes a_{22} = \sum a_1 \otimes a_{211} \otimes a_{212} \otimes a_{22}, \tag{2.9}$$

where $\delta(a_{21}) = \Sigma a_{211} \otimes a_{212}$ with all factors homogeneous.

Now define a K-linear map L: $A \otimes A \otimes A \otimes A \to A \otimes A \otimes A \otimes A$ by

$$L(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = c^{\chi(|a_2|+|a_3|,|a_4|)}(a_1 \otimes a_2 \otimes a_3 \otimes a_4)$$

for all a_1, a_2, a_3, a_4 homogeneous.

By applying the K-linear map $O = (m \otimes \mathrm{id})(\mathrm{id} \otimes T)(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes s \otimes s)L$ to both sides of (2.9), and applying **R3**, we get

$$\begin{aligned}
\sum c^{\chi(|a_{12}|+|a_{21}|,|a_{22}|)} a_{11} s(a_{22}) \otimes a_{12} s(a_{21})
&= \sum c^{\chi(|a_{211}|+|a_{212}|,|a_{22}|)} a_1 s(a_{22}) \otimes a_{211} s(a_{212}) \\
&= \sum c^{\chi(|a_{21}|,|a_{22}|)} a_1 s(a_{22}) \otimes a_{211} s(a_{212}) \\
&= \sum c^{\chi(|a_2|-|a_{22}|,|a_{22}|)} a_1 s(a_{22}) \otimes a_{211} s(a_{212}) \\
&= \sum c^{\chi(|a_2|-|a_{22}|,|a_{22}|)} a_1 s(a_{22}) \otimes \varepsilon(a_{21}) \\
&= \sum c^{\chi(|a_2|-|a_{22}|,|a_{22}|)} a_1 s(\varepsilon(a_{21}) a_{22}) \otimes 1.
\end{aligned}$$

We claim that

$$c^{\chi(|a_2|-|a_{22}|,|a_{22}|)} a_1 s(\varepsilon(a_{21}) a_{22}) \otimes 1 = a_1 s(\varepsilon(a_{21}) a_{22}) \otimes 1.$$

In fact, if $|a_{21}| \neq 0$, then $\varepsilon(a_{21}) = 0$, and hence the both sides are 0; if $|a_{21}| = 0$, then $|a_2| - |a_{22}| = 0$, and hence the claim follows.

In this way, we see that

$$\begin{aligned}
\sum c^{\chi(|a_{12}|+|a_{21}|,|a_{22}|)} a_{11} s(a_{22}) \otimes a_{12} s(a_{21})
&= \sum c^{\chi(|a_2|-|a_{22}|,|a_{22}|)} a_2 s(\varepsilon(a_{21}) a_{22}) \otimes 1 \\
&= \sum a_1 s(\varepsilon(a_{21}) a_{22}) \otimes 1 \\
&= \sum a_1 s(a_2) \otimes 1 \\
&= \varepsilon(a) \otimes 1 \\
&= (e \otimes e) \varepsilon(a),
\end{aligned}$$

where we have used **R3** and the counit property.

Altogether, we have proved that $(\delta s) \bigstar \delta = \delta \bigstar (T(s \otimes s)\delta_\chi)$.

(v) By Lemma 2.5, ${}_{\chi^T}A = (A, {}_{\chi^T}\delta, \varepsilon)$ is a K-coalgebra. In order to prove that ${}_{\chi^T}A = (A, m, e, {}_{\chi^T}\delta, \varepsilon)$ is a $(K, c, I, -\chi^T)$-bialgebra, it suffices to prove ${}_{\chi^T}\delta(ab) = {}_{\chi^T}\delta(a) * {}_{\chi^T}\delta(b)$ for all $a, b \in A$, where $*$ denotes the multiplication in $(A \otimes A)_{-\chi^T}$. This is straightforward by using the fact that

$\delta\colon A \to (A \otimes A)_\chi$ is an algebra homomorphism. It follows from (i) that ${}_{\chi^T}A = (A, m, e, {}_{\chi^T}\delta, \varepsilon)$ is a $(K, c, I, -\chi^T)$-Hopf algebra, say, with antipode s'.

For $a \in A_d$, use induction on d to prove that $s's(a) = ss'(a) = a$. This is clear for $a \in A_0$. By Lemma 2.2 we have $\delta(a) = a \otimes 1 + 1 \otimes a + \sum_{x+y=d;\, x,y \neq 0} c_x \otimes c'_y$ with $c_x \in A_x$, $c'_y \in A_y$. Then by (2.3) we have

$$ {}_{\chi^T}\delta(a) = a \otimes 1 + 1 \otimes a + \sum_{x+y=d;\, x,y\neq 0} c^{-\chi(y,x)} c'_y \otimes c_x. $$

By using $m(\mathrm{id} \otimes s)\delta(a) = e\varepsilon(a) = 0$ and $m(\mathrm{id} \otimes s')_{\chi^T}\delta(a) = e\varepsilon(a) = 0$ we get

$$ s(a) = -a - \sum_{x+y=d;\, x,y\neq 0} c_x s(c'_y), $$
$$ s'(a) = -a - \sum_{x+y=d;\, x,y\neq 0} c^{-\chi(y,x)} c'_y s'(c_x). $$

By (iii), $s'\colon A \to A_{-\chi}$ is an algebra anti-homomorphism; it follows from induction that

$$ \begin{aligned} s's(a) &= -s'(a) - \sum_{x+y=d;\, x,y\neq 0} s's(c_y) * s'(c_x) \\ &= a + \sum_{x+y=d;\, x,y\neq 0} c^{-\chi(y,x)} c'_y s'(c_x) - \sum_{x+y=d;\, x,y\neq 0} c^{-\chi(y,x)} c'_y s'(c_x) \\ &= a. \end{aligned} $$

Dually, we have $ss'(a) = a$.

This completes the proof. ∎

In general, it is not always easy to verify a given map $s\colon A \to A$ is the antipode for a χ-bialgebra A, but it should be simpler to check (2.7) only on generators of A. Thus, it is convenient to have the following

LEMMA 2.11. *Let A be a χ-bialgebra and let $s\colon A \to A_{\chi^T}$ be an $\mathbb{N}_0 I$-graded algebra anti-homomorphism. Assume that A is generated as an algebra by a subset X consisting of homogeneous elements of A, such that* (2.7) *holds for all $a \in X$. Then s is the antipode of χ-Hopf algebra A.*

Proof. By assumption, it suffices to prove that if $m(\mathrm{id} \otimes s)\delta(a) = \varepsilon(a)$ and $m(\mathrm{id} \otimes s)\delta(b) = \varepsilon(b)$, then $m(\mathrm{id} \otimes s)\delta(ab) = \varepsilon(ab)$, where a, b are homogeneous elements in A.

Let $\delta(a) = \sum a_1 \otimes a_2$ and $\delta(b) = \sum b_1 \otimes b_2$ with all factors homogeneous. Then by **R2** and (2.5) we have

$$\delta(ab) = \sum c^{\chi(|a_2|,|b_1|)} a_1 b_1 \otimes a_2 b_2,$$

and hence

$$\begin{aligned} m(\mathrm{id} \otimes s)\delta(ab) &= \sum c^{\chi(|a_2|,|b_1|)} a_1 b_1 s(a_2 b_2) \\ &= \sum c^{\chi(|a_2|,|b_1|)+\chi^T(|b_2|,|a_2|)} a_1 b_1 s(b_2) s(a_2) \\ &= \sum c^{\chi(|a_2|,|b|)} a_1 b_1 s(b_2) s(a_2) \\ &= \sum c^{\chi(|a_2|,|b|)} a_1 s(a_2) \varepsilon(b). \end{aligned}$$

It follows that if $b \notin A_0$, then $\varepsilon(b) = 0$ and $m(\mathrm{id} \otimes s)\delta(ab) = 0 = \varepsilon(ab)$, and if $b \in A_0$, then

$$\begin{aligned} m(\mathrm{id} \otimes s)\delta(ab) &= \sum a_1 s(a_2)\varepsilon(b) = m(\mathrm{id} \otimes s)\delta(a)\varepsilon(b) \\ &= \varepsilon(a)\varepsilon(b) = \varepsilon(ab). \end{aligned}$$

∎

EXAMPLE 2.12 [L, p. 2]. Let K, c, I, χ be as in 2.8. Consider Lusztig's algebra $'\mathbf{F}$, which is by definition the free k-algebra with generators θ_i, $i \in I$. Then $'\mathbf{F}$ is an $\mathbb{N}_0 I$-graded algebra with $'\mathbf{F} = \oplus_{x \in \mathbb{N}_0 I} {'\mathbf{F}_x}$, where for $x = (x_i)_{i \in I} \in \mathbb{N}_0 I$, $'\mathbf{F}_x$ is the K-space with basis the monomials $\theta_{i_1}, \dots, \theta_{i_l}$, where $l = \sum_{i \in I} x_i$, such that for any $i \in I$, the number of occurrences of i in the sequence $i_1, \dots, i_l$ is equal to x_i. In particular, $'\mathbf{F}_0 = K$. Let δ: $'\mathbf{F} \to ('\mathbf{F} \otimes {'\mathbf{F}})_\chi$ be the unique algebra homomorphism such that $\delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$, $i \in I$, and $\varepsilon: {'\mathbf{F}} \to K$ be the unique algebra homomorphism such that $\varepsilon(1) = 1$, $\varepsilon(\theta_i) = 0$. Then $'\mathbf{F}$ is a χ-bialgebra and hence a χ-Hopf algebra by Theorem 2.10(i). Let $s: {'\mathbf{F}} \to {'\mathbf{F}_{\chi^T}}$ be the unique algebra anti-homomorphism such that $s(1) = 1$ and $s(\theta_i) = -\theta_i$ for all $i \in I$. It follows from Lemma 2.11 that s is the antipode of the χ-Hopf algebra $'\mathbf{F}$.

EXAMPLE 2.13. In his study of quantum shuffle algebras, Rosso [M] has given a class of twisted bialgebras. In [Ro, Proposition 12] Rosso defines a twisted bialgebra structure on the tensor space $T(V)$ with algebra generators θ_i $(i \in I)$ (Rosso uses different notation) satisfying the relation $\theta_i \theta_j = q_{ij} \theta_j \theta_i$, where q_{ij} are arbitrary non-zero elements of K, and $T(V) \otimes T(V)$ has a multiplication rule as Lusztig's, with q_{ij} replacing $c^{\chi(i,j)}$ in (2.5) in 2.6.

Remark 2.14. We are grateful to the following observation by the referee: Note that any χ-Hopf algebra A can be embedded in an "untwisted" Hopf algebra $T \otimes A$, where T is the group algebra over K of the additive group $\mathbb{Z}I$, and $T \otimes A$ is a "twisted product" of T and A. This twist is so designed that it "cancels" the twist in $(A \otimes A)_\chi$, with the result that $T \otimes A$ is a Hopf algebra in the usual sense. This construction of Hopf algebra $T \otimes A$ in special cases has been used by Lusztig [L, p. 19] for the algebra **f** and by Xiao [X, p. 122] for the Ringel–Hall algebra, and it also works in general; see the Appendix at the end of this paper by the referee.

3. χ-HOPF ALGEBRA STRUCTURE ON TWISTED RINGEL ALGEBRAS

The aim of this section is to study the properties of a twisted Ringel–Hall algebra and Ringel's composition algebra, from the point of view of a χ-Hopf algebra structure.

3.1. We briefly recall some basic points on the twisted Ringel–Hall algebras and Ringel's composition algebras.

Let k be a finite field with $|k| = q_k$. Set $v_k = \sqrt{q}_k$. Let Λ be a finite-dimensional, hereditary k-algebra with all simple Λ-modules $S(1), \dots, S(n)$, up to isomorphism. Denote by Λ-mod the category of finite-dimensional left Λ-modules, which is exactly the category of Λ-modules with finitely many elements. The Grothendieck group $K_0(\Lambda)$ of all finite Λ-modules modulo short exact sequences can be identified with $\mathbb{Z}^n$, such that the image of $S(i)$ in it is the ith coordinate vector. For $M \in \Lambda$-mod, denote its isoclass by $[M]$ and its image in $K_0(\Lambda)$ by $\mathbf{dim}\, M$ which is called the dimension vector of M. Let $\mathscr{P}$ denote the set of isoclasses of all modules in Λ-mod, $\mathscr{P}_1 = \mathscr{P} \setminus \{[0]\}$.

Given $M, N \in \Lambda$-mod, let

$$\langle M, N \rangle = \dim_k \operatorname{Hom}_\Lambda(M, N) - \dim_k \operatorname{Ext}^1_\Lambda(M, N). \tag{3.1}$$

Since Λ is hereditary, the integer $\langle M, N \rangle$ depends only on the dimension vectors $\mathbf{dim}\, M$ and $\mathbf{dim}\, N$, not on M and N themselves. So $\langle -, - \rangle$ induces a bilinear form on $\mathbb{Z}^n$, which will be called the Ringel form. Denote its symmetrization by $(-, -)$. Thus,

$$(M, N) = \langle M, N \rangle + \langle N, M \rangle. \tag{3.2}$$

Given $M, N_1, \dots, N_t \in \Lambda$-mod, let $g^M_{N_1, \dots, N_t}$ be the number of filtrations

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$$

of M such that $M_{i-1}/M_i \cong N_i$ for $1 \le i \le t$. In particular, $g_N^M = 1$ if $M \cong N$ and $g_N^M = 0$ of $M \not\cong N$.

By definition (see [R4]) the twisted Ringel–Hall algebra $\mathscr{H}(\Lambda)$ of Λ is the $\mathbb{Q}(v_k)$-space with basis $\mathscr{P}$ and a $\mathbb{Q}(v_k)$-bilinear multiplication given by

$$[M][N] = \sum_{[L] \in \mathscr{P}} v_k^{\langle M, N\rangle} g_{M,N}^L [L]. \tag{3.3}$$

Then $\mathscr{H}(\Lambda)$ is an $\mathbb{N}_0^n$-graded, associative $\mathbb{Q}(v_k)$-algebra with unit [0], where for $x \in \mathbb{N}_0^n$, $\mathscr{H}(\Lambda)_x$ is the $\mathbb{Q}(v_k)$-space with basis $\{[M] \mid \mathbf{dim}\, M = x\}$. In particular, $\mathscr{H}(\Lambda)_0 = \mathbb{Q}(v_k)$.

Note that here we use the twisted multiplication for $\mathscr{H}(\Lambda)$; for the original one see [R2, M] (but for discrete valuation rings).

Let a_M denote the order of automorphism group of Λ-module M. Define the $\mathbb{Q}(v_k)$-linear map $\delta\colon \mathscr{H}(\Lambda) \to \mathscr{H}(\Lambda) \otimes \mathscr{H}(\Lambda)$ by (see [G1])

$$\delta([L]) = \sum_{[M],[N] \in \mathscr{P}} v_k^{\langle M, N\rangle} \frac{a_M a_N}{a_L} g_{M,N}^L ([M] \otimes [N]). \tag{3.4}$$

Define the $\mathbb{Q}(v_k)$-linear map $\varepsilon\colon \mathscr{H}(\Lambda) \to \mathbb{Q}(v_k)$ by (see [R6])

$$\varepsilon([M]) = \begin{cases} 1, & [M] = [0] \\ 0, & [M] \ne [0]. \end{cases} \tag{3.5}$$

Set χ to be the symmetrization $(-,-)$ of the Ringel form $\langle -,- \rangle$, and set I to be a set of n elements, where $K_0(\Lambda) = \mathbb{Z}^n$. Then by [R6, p. 212; G1, Theorem 1] and Theorem 2.10 we have

THEOREM (Ringel–Green). *$\mathscr{H}(\Lambda) = (\mathscr{H}(\Lambda), \delta, \varepsilon)$ is a $(\mathbb{Q}(v_k), v_k, I, \chi)$-bialgebra and hence a $(\mathbb{Q}(v_k), v_k, I, \chi)$-Hopf algebra with antipode s, where the multiplication on $\mathscr{H}(\Lambda) \otimes \mathscr{H}(\Lambda)$ is given by*

$$([M] \otimes [N]) * ([M'] \otimes [N']) = v_k^{(N, M')}([M][M'] \otimes [N][N']) \tag{3.6}$$

for $M, N, M', N' \in \Lambda$-mod.

Moreover, $s\colon \mathscr{H}(\Lambda) \to \mathscr{H}(\Lambda)_\chi$ is an $\mathbb{N}_0^n$-graded, $\mathbb{Q}(v_k)$-algebra anti-isomorphism, where the multiplication on $\mathscr{H}(\Lambda)_\chi$ is given by

$$[M] * [N] = v_k^{(M,N)}[M][N] = \sum_{[L] \in \mathscr{P}} v_k^{2\langle M, N\rangle + \langle N, M\rangle} g_{M,N}^L [L] \tag{3.7}$$

for all $[M],[N] \in \mathscr{P}$, and $s\colon \mathscr{H}(\Lambda)_\chi \to \mathscr{H}(\Lambda)$ is an $\mathbb{N}_0^n$-graded, $\mathbb{Q}(v_k)$-coalgebra anti-isomorphism, where the comultiplication on $\mathscr{H}(\Lambda)_\chi$ is given by

$$\delta_\chi([L]) = \sum_{[M],[N] \in \mathscr{P}} v_k^{2\langle M, N\rangle + \langle N, M\rangle} \frac{a_M a_N}{a_L} g_{M,N}^L ([M] \otimes [N]) \tag{3.8}$$

for all $[L] \in \mathscr{P}$.

3.2. By definition [R3, R5], the composition algebra $\mathscr{C}(\Lambda)$ is the subalgebra of $\mathscr{H}(\Lambda)$ generated by $[S(i)]$, $1 \leq i \leq n$. Thus, $\mathscr{C}(\Lambda) = \oplus_{x \in \mathbb{N}_0^n} \mathscr{C}(\Lambda)_x$ is an $\mathbb{N}_0^n$-graded algebra, where for $x = (x_i) \in \mathbb{N}_0^n$, $\mathscr{C}(\Lambda)_x$ is the $\mathbb{Q}(v_k)$-space spanned by all monomials $[S(i_1)] \cdots [S(i_l)]$ with $l = \Sigma_{1 \leq i \leq n} x_i$, such that the number of occurrences of i in the sequence $i_1, \dots, i_l$ equals x_i for $1 \leq i \leq n$. Since $\delta([S(i)]) = [S(i)] \otimes 1 + 1 \otimes [S(i)]$, it follows that $\mathscr{C}(\Lambda)$ is also a χ-bialgebra. Since s is an algebra anti-isomorphism with $s([S(i)]) = -[S(i)]$, it follows that $s(\mathscr{C}(\Lambda)) = \mathscr{C}(\Lambda)$. Thus we have

COROLLARY. *The composition algebra $\mathscr{C}(\Lambda)$ is a χ-Hopf subalgebra of $\mathscr{H}(\Lambda)$ with invertible antipode s. The assertions in Theorem* 3.1 *also hold for $\mathscr{C}(\Lambda)$. In particular, we have*

$$s([S(i)][S(j)]) = v_k^{\langle S(i), S(j)\rangle}[S(j)][S(i)] \tag{3.9}$$

for all $1 \leq i, j \leq n$.

LEMMA 3.3. *The antipode s of $\mathscr{H}(\Lambda)$ is given by $s([0]) = [0]$, and for $[M] \in \mathscr{P}_1$ by*

$$s([M]) = \sum_{[L] \in \mathscr{P}_1} \sum_{t \geq 1} (-1)^t \times \sum_{[N_1], \dots, [N_t] \in \mathscr{P}_1} v_k^{2\Sigma_{i<j}\langle N_i, N_j\rangle} \frac{a_{N_1} \cdots a_{N_t}}{a_M} g^M_{N_1, \dots, N_t} g^L_{N_1, \dots, N_t}[L], \tag{3.10}$$

with the inverse s^{-1} given by $s^{-1}([0]) = [0]$, and for $[M] \in \mathscr{P}_1$ by

$$s^{-1}([M]) = \sum_{[L] \in \mathscr{P}_1} \sum_{t \geq 1} (-1)^t \times \sum_{[N_1], \dots, [N_t] \in \mathscr{P}_1} \frac{a_{N_1} \cdots a_{N_t}}{a_M} g^M_{N_t, \dots, N_1} g^L_{N_1, \dots, N_t}[L]. \tag{3.11}$$

Proof. We have seen that $\mathscr{H}(\Lambda)$ is a $(\mathbb{Q}(v_k), v_k, I, \chi)$-Hopf algebra, with antipode s. From the "extended" (twisted) Ringel–Hall algebra in the sense of [X], which is a Hopf algebra with antipode, say s'. By Theorem A in the Appendix by the referee we have the relation between s and s', and then the formulae follow from [X, Theorem 4.5(c), p. 123]. ∎

3.4. Green [G1] has also introduced a $\mathbb{Q}(v_k)$-valued (or $\mathbb{R}$-valued, where $\mathbb{R}$ is the field of real numbers), positive-definite, symmetric, bilinear form $(-, -)$ on $\mathscr{H}(\Lambda)$. Let $M, N \in \Lambda$-mod. Then $(-, -)$: $\mathscr{H}(\Lambda) \times \mathscr{H}(\Lambda)$

$\to \mathbb{Q}(v_k)$ is the $\mathbb{Q}(v_k)$-bilinear form defined as follows:

$$([M],[N]) = \begin{cases} \dfrac{|M|}{a_M}, & [M]=[N] \\ 0, & [M] \neq [N]. \end{cases} \tag{3.12}$$

Here, we use Ringel's modification on the values of $(-,-)$ such that it coincides with Lusztig's form on **f**; see [R6]. Note that we will use $\mathbb{R}$ instead of $\mathbb{Q}(v_k)$, if the Euclidean space $\mathscr{H}(\Lambda)_x$ is emphasized.

One can distinguish the form $(-,-)$ on $\mathscr{H}(\Lambda)$ defined above from the symmetrization $(-,-)$ of the Ringel form in context.

Note that the comultiplication δ of $\mathscr{H}(\Lambda)$ is adjoint to the multiplication under the bilinear form $(-,-)$ on $\mathscr{H}(\Lambda)$, i.e.,

$$(a, bc) = (\delta(a), b \otimes c) \tag{3.13}$$

for $a, b, c \in \mathscr{H}(\Lambda)$, where the form $(-,-)$ on $\mathscr{H}(\Lambda) \times \mathscr{H}(\Lambda)$ is defined by

$$(a_1 \otimes a_2, b_1 \otimes b_2) = (a_1, b_1)(a_2, b_2)$$

for all $a_1, a_2, b_1, b_2 \in \mathscr{H}(\Lambda)$.

The following theorem shows that the antipode s of $\mathscr{H}(\Lambda)$ is self-adjoint under the bilinear form $(-,-)$ on $\mathscr{H}(\Lambda)$.

THEOREM 3.5. *Let $(-,-)$ be the bilinear form on $\mathscr{H}(\Lambda)$. Then for all $a, b \in \mathscr{H}(\Lambda)$ we have*

$$(s(a), b) = (a, s(b)). \tag{3.14}$$

Proof. Since $(-,-)$ respects the grading, i.e., $(\mathscr{H}(\Lambda)_x, \mathscr{H}(\Lambda)_y) = 0$ for $x \neq y$, and s preserves the grading, it follows that it suffices to consider the case $a = [M]$, $b = [N]$ with $\mathbf{dim}\, M = \mathbf{dim}\, N \neq 0$. By Lemma 3.3 we have

$$\begin{aligned} (s([M]),[N]) &= \sum_{[L]\in\mathscr{P}_1} \sum_{t\geq 1} (-1)^t \sum_{[N_1],\dots,[N_t]\in\mathscr{P}_1} v_k^{2\sum_{i<j}\langle N_i, N_j\rangle} \\ &\quad \times \frac{a_{N_1}\cdots a_{N_t}}{a_M} g^M_{N_1,\dots,N_t} g^L_{N_1,\dots,N_t}([L],[N]) \\ &= \sum_{t\geq 1} (-1)^t \sum_{[N_1],\dots,[N_t]\in\mathscr{P}_1} v_k^{2\sum_{i<j}\langle N_i, N_j\rangle} \\ &\quad \times \frac{a_{N_1}\cdots a_{N_t}|N|}{a_M a_N} g^M_{N_1,\dots,N_t} g^N_{N_1,\dots,N_t} \end{aligned}$$

and

$$([M], s([N])) = \sum_{t\geq 1} (-1)^t \sum_{[N_1],\ldots,[N_t]\in\mathscr{P}_1} v_k^{2\sum_{i<j}\langle N_i, N_j\rangle} \times \frac{a_{N_1}\cdots a_{N_t}|M|}{a_N a_M} g^N_{N_1,\ldots,N_t} g^M_{N_1,\ldots,N_t}.$$

Then the assertion follows from $|M| = |N|$, since $\mathbf{dim}\, M = \mathbf{dim}\, N$. ∎

3.6. Let $0 \neq x \in \mathbb{N}_0^n$. Denote by

$$A_x = \sum_{y+z=x;\, y,z\neq 0} \mathscr{H}(\Lambda)_y \mathscr{H}(\Lambda)_z \subseteq \mathscr{H}(\Lambda)_x.$$

Since the bilinear form $(-,-)$ on $\mathscr{H}(\Lambda)$ is positive definite and every homogeneous component $\mathscr{H}(\Lambda)_x$ is finite dimensional, we have the orthogonal complement B_x of A_x in $\mathscr{H}(\Lambda)_x$, i.e.,

$$(A_x, B_x) = 0, \qquad A_x \oplus B_x = \mathscr{H}(\Lambda)_x.$$

Choose an orthogonal basis of B_x, and denote by $(\theta_i)_{i\in I}$ the union of these bases over $0 \neq x \in \mathbb{N}_0^n$. The following observation is due to Sevenhant and Van den Bergh [SV]

LEMMA. *With the notations as above, $(\theta_i)_{i\in I}$ is a set of homogeneous generators of $\mathscr{H}(\Lambda)$, and every θ_i is δ-primitive, i.e., $\delta(\theta_i) = 1 \otimes \theta_i + \theta_i \otimes 1$.*

Proof. It is clear by construction and induction that $(\theta_i)_{i\in I}$ is a set of generators of $\mathscr{H}(\Lambda)$. For the convenience of the reader, we quoted the following argument from [SV].

Choose a homogeneous orthogonal basis $(f_j)_{j\in J}$ for $\mathscr{H}(\Lambda)$, and assume that $(\theta_i)_{i\in I} \subset (f_j)_{j\in J}$. Write

$$\delta(\theta_i) = \sum_{j,l} c_{j,l} f_j \otimes f_l$$

for $c_{j,l} \in \mathbb{R}$. Then by (3.13) we have

$$(\theta_i, f_m f_t) = \sum_{j,l} c_{j,l}(f_j, f_m)(f_l, f_t) = c_{mt}.$$

Thus, if θ_i and $f_m f_t$ are in different homogeneous components of $\mathscr{H}(\Lambda)$, then $c_{mt} = 0$; and if θ_i and $f_m f_t$ are in the same homogeneous component $\mathscr{H}(\Lambda)_x$, then by the definition of $(\theta_i)_{i\in I}$ we also have $c_{mt} = 0$, unless $f_m = 1$ and $f_t = \theta_i$, or $f_t = 1$ and $f_m = \theta_i$. This completes the proof. ∎

THEOREM 3.7. *Keep the notations in* 3.6. *Then we have*

(i) *The set of δ-primitive elements of $\mathscr{H}(\Lambda)$ is exactly*

$$\bigoplus_{x \in \mathbb{N}_0^n} B_x;$$

i.e., $(\theta_i)_{i \in I}$ *is a basis of the space of δ-primitive elements of $\mathscr{H}(\Lambda)$.*

(ii) $\mathscr{C}(\Lambda) = \mathscr{H}(\Lambda)$ *if and only if the set of δ-primitive elements of $\mathscr{H}(\Lambda)$ is exactly the space with* $[S(i)]$, $1 \le i \le n$, *as a basis.*

Proof. (i) Let $a = \sum_{x \in \mathbb{N}_0^n} a_x$ be a δ-primitive element of $\mathscr{H}(\Lambda)$ with $0 \neq a_x \in \mathscr{H}(\Lambda)_x$. By Lemma 3.6 it suffices to prove the $a_x \in B_x$.

First, we claim that all homogeneous components a_x are δ-primitive elements. In fact, by Lemma 2.2 we have $\delta(a_x) = a_x \otimes 1 + 1 \otimes a_x + \sum_{y+z=x;\, y,z \neq 0} a_{x,y} \otimes a'_{x,z}$, where $a_{x,y} \in \mathscr{H}(\Lambda)_y$, $a'_{x,z} \in \mathscr{H}(\Lambda)_z$, and hence we have

$$\begin{aligned}\sum_x (1 \otimes a_x + a_x \otimes 1) &= 1 \otimes a + a \otimes 1 \\ &= \delta(a) = \sum_x \delta(a_x) \\ &= \sum_x \Big(1 \otimes a_x + a_x \otimes 1 + \sum_{y+z=x;\, y,z \neq 0} a_{x,y} \otimes a'_{x,z}\Big),\end{aligned}$$

and hence the claim follows by comparing the homogeneous components of degree (y, z).

Now, assume that some $a_x \notin B_x$. Then $a_x = b_x + a'_x$ with $b_x \in B_x$ and $0 \neq a'_x \in A_x$, and hence a'_x is also a δ-primitive element. By definition of A_x we can write $a'_x = \sum_{y+z=x;\, y,z \neq 0} b_y b'_z$ with $b_y \in A_y$, $b'_z \in A_z$. Then by (3.13) we obtain the contradiction

$$\begin{aligned} 0 &\neq (a'_x, a'_x) \\ &= \sum_{y+z=x;\, y,z \neq 0} (a'_x, b_y b'_z) \\ &= \sum_{y+z=x;\, y,z \neq 0} (1 \otimes a'_x + a'_x \otimes 1, b_y \otimes b'_z) \\ &= 0. \end{aligned}$$

This proves (i).

(ii) Note that $\mathscr{C}(\Lambda) = \mathscr{H}(\Lambda)$ if and only if $\mathscr{C}(\Lambda)_x = \mathscr{H}(\Lambda)_x$ for all $x \in \mathbb{N}_0^n$. Since $\mathscr{C}(\Lambda)_x = \sum_{y+z=x\, y,z \neq 0} \mathscr{C}(\Lambda)_y \mathscr{C}(\Lambda)_z$ for $x \neq \mathbf{dim}\, S(i)$, $1 \le i \le n$, it follows that $\mathscr{C}(\Lambda) = \mathscr{H}(\Lambda)$ if and only if $B_x = 0$ for $x \neq \mathbf{dim}\, S(i)$, $1 \le i \le n$, and by (i) if and only if the set of δ-primitive elements of $\mathscr{H}(\Lambda)$ is exactly the spaces with $[S(i)]$, $1 \le i \le n$, as a basis. ∎

For more information on comparing $\mathscr{C}(\Lambda)$ and $\mathscr{H}(\Lambda)$ see [R5, Z3].

4. GREEN'S CATEGORIES AND POLYNOMIALS

4.1. Throughout this section, let K be a field, let c be a non-zero element in K, and let $(I, \cdot)$ be a datum. Recall that a pair $(I, \cdot)$ is called a datum if I is a set and $\cdot: \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ is a symmetric, bilinear form.

In his papers [G1, G2], Green has introduced the following interesting category $\mathscr{L}(K, c, I, \cdot)$, the aim of this section is to give some more properties of this category.

By definition [G1, G2], an object L in the category $\mathscr{L}(K, c, I, \cdot)$ is a K-algebra satisfying the following properties:

L1. $L = \oplus_{x \in \mathbb{N}_0 I} L_x$ with $L_0 = K$ is an $\mathbb{N}_0 I$-graded, associative K-algebra generated by elements $u_i \in L_i$, $i \in I$. We call u_i, $i \in I$, the generators of L.

L2. There is an algebra homomorphism $\delta: L \to (L \otimes L)_\chi$, where the algebra structure on $(L \otimes L)_\chi$ is given by Lusztig's rule, i.e., the formula (2.5) in 2.6 with $\chi = \cdot$, such that all u_i are δ-primitive elements.

L3. There is a symmetric bilinear form $(-,-): L \times L \to K$ such that

(i) $(-,-)$ respects the grading, i.e., $(L_x, L_y) = 0$ for $x \neq y$ in $\mathbb{N}_0 I$.

(ii) $(1,1) = 1$; $(u_i, u_i) \neq 0$ for $i \in I$.

(iii) For $a, b, c \in L$ there holds

$$(a, bc) = (\delta(a), b \otimes c)$$

where the bilinear form $(-,-)$ on $L \times L$ is defined by

$$(a_1 \otimes a_2, b_1 \otimes b_2) = (a_1, b_1)(a_2, b_2).$$

The bilinear form on L also will be called the Lusztig form.

A morphism between objects L, L' of $\mathscr{L}(K, c, I, \cdot)$ is a K-algebra homomorphism $\psi: L \to L'$ sending the generators u_i of L to the generators u_i' of L', $i \in I$.

4.2. *Notations*. Let $x = (x_i)_{i \in I} \in \mathbb{N}_0 I$ with length $l = \sum_{i \in I} x_i$. Denote by $I(x)$ the set of all vectors $a = (a_1, \ldots, a_l) \in I^l$ with weight x, i.e., with the property

$$\left|\{t \mid 1 \leq t \leq l, a_t = i\}\right| = x_i, \qquad i \in I.$$

Then

$$|I(x)| = \frac{(\sum_{i \in I} x_i)!}{\prod_{i \in I} x_i!}.$$

Let $L \in \mathscr{L}(K, c, I, \cdot)$ with generators u_i, $i \in I$. For $a = (a_1, \dots, a_l) \in I(x)$, define

$$u_a = \begin{cases} u_{a_1} \cdots u_{a_l}, & l > 0 \\ 1, & l = 0. \end{cases} \tag{4.1}$$

Define

$$\sigma(a) = (a_l, \dots, a_1) \in I(x). \tag{4.2}$$

LEMMA 4.3. *Assume that L is a $(K, c, I, \cdot)$-Hopf algebra with antipode s and satisfies* **L1** *with generators u_i, $i \in I$. Let $x = (x_i)_{i \in I} \in \mathbb{N}_0 I$ with length l, and $a = (a_1, \dots, a_l) \in I(x)$. Then*

$$s(u_a) = \lambda(x) u_{\sigma(a)}, \tag{4.3}$$

where

$$\lambda(x) = (-1)^l c^{\sum_{i \neq j, \in I} x_i x_j (i \cdot j/2) + \sum_{i \in I} x_i (x_i - 1) i \cdot i/2}, \tag{4.4}$$

and $\sigma(a)$ is as in (4.2).

In particular, $s(u_i) = -u_i$, $i \in I$.

Proof. Set $\chi = \cdot$. According to Theorem 2.10, the antipode s: $L \to L_\chi$ is an $\mathbb{N}_0 I$-graded algebra anti-homomorphism with $s(u_i) = -u_i$, where the multiplication $*$ of L_χ is given by the rule (2.1) in 2.3. It follows that

$$s(u_a) = (-1)^l u_{a_l} * \cdots * u_{a_1} = (-1)^l c^{\sum_{w > t} a_t \cdot a_w} u_{a_l} \cdots u_{a_1} = \lambda(x) u_{\sigma(a)},$$

where by easy calculations we have

$$\lambda(x) = (-1)^l c^{\sum_{i \neq j, \in I} x_i x_j (i \cdot j/2) + \sum_{i \in I} x_i (x_i - 1) i \cdot i/2},$$

where the symmetry of $\cdot$ has been used. ∎

Remark. By the formula (4.4) we see that $\lambda(x)$ depends only on $x \in \mathbb{Z} I$, not on $a \in I(x)$. This is no longer true if $\cdot$ is not symmetric. We shall use this fact in the proof of Theorem 4.7 below, and this is the reason for the assumption of the symmetry of $\cdot$ in this section.

The following theorem gives the main properties of an object in the category $\mathscr{L}(K, c, I, \cdot)$ which relates to its antipode s.

THEOREM 4.4. *Let L be an object in $\mathscr{L}(K, c, I, \cdot)$. Then*

(i) *L becomes a $(K, c, I, \cdot)$-Hopf algebra, where the counit ε of L is the projection onto $L_0 = K$.*

(ii) *The antipode s of L is self-adjoint under the symmetric bilinear form $(-, -)$ on L, i.e.,*

$$(s(x), y) = (x, s(y)), \qquad \forall x, y \in L. \tag{4.5}$$

(iii) *Any morphism* ψ *between objects* $L, L' \in \mathscr{L}(K, c, I, \cdot)$ *is a* $(K, c, I, \cdot)$*-Hopf algebra homomorphism* (*i.e.*, $\psi\colon L \to L'$ *is simultaneously a* K*-algebra homomorphism and a* K*-coalgebra homomorphism such that* $\psi s = s'\psi$, *where* s' *is the antipode of* L').

(iv) *Let* $\mathscr{I}$ *be the two-sided ideal* $\{a \in L \mid (a, x) = 0, \forall x \in L\}$ *of* L. *Then* $s(\mathscr{I}) = \mathscr{I}$, *and* $L/\mathscr{I} \in \mathscr{L}(K, c, I, \cdot)$ *with the antipode induced by* s.

(v) *Let* δ *be the comultiplication of* L. *If the bilinear form* $(-, -)$ *on* L *is positive definite, then the set of* δ*-primitive elements is* $\oplus_{i \in I} L_i$.

Proof. (i) Let δ be the K-map for L given in **L2**. Then δ is coassociative, and then by **L1** and **L2** one sees that $L = (L, \delta, \varepsilon)$ is a $(K, c, I, \cdot)$-bialgebra, and hence from Theorem 2.10(i) the assertion follows.

(ii) Since $(-, -)$ respects the grading, and s is a graded map, it suffices to prove (4.5) for $x, y \in L_\mu$ being monomials. Use induction on μ. It is clear for $\mu = 0$ and $\mu = i \in I$. Assume that $x = x'x''$, $y = y'y''$ with all factors being monomials not in $L_0 = K$. Let $\delta(x') = \sum x'_1 \otimes x'_2$, $\delta(x'') = \sum x''_1 \otimes x''_2$. Then

$$\delta(x) = \sum c^{|x'_2| \cdot |x''_1|} x'_1 x''_1 \otimes x'_2 x''_2.$$

Set $\chi = \cdot$. Then

$$\delta_\chi(x) = \sum c^{|x'_2| \cdot |x''_1| + |x'_1 x''_1| \cdot |x'_2 x''_2|} x'_1 x''_1 \otimes x'_2 x''_2.$$

By Theorem 2.10(iv) we have

$$\begin{aligned}
(s(x), y) &= (\delta s(x), y' \otimes y'') \\
&= \big(T(s \otimes s)\delta_\chi(x), y' \otimes y''\big) \\
&= \sum c^{|x'_2| \cdot |x''_1| + |x'_1 x''_1| \cdot |x'_2 x''_2|} \big(s(x'_2 x''_2) \otimes s(x'_1 x''_1), y' \otimes y''\big) \\
&= \sum c^{|x'_2| \cdot |x''_1| + |x'_1 x''_1| \cdot |x'_2 x''_2|} \big(s(x'_2 x''_2), y'\big)\big(s(x'_1 x''_1), y''\big) \\
&= \sum c^{|x'_2| \cdot |x''_1| + |x'_1 x''_1| \cdot |x'_2 x''_2|} \big(x'_2 x''_2, s(y')\big)\big(x'_1 x''_1, s(y'')\big) \\
&= \sum c^{|x'_2| \cdot |x''_1| + |y'| \cdot |y''|} \big(x'_1 x''_1, s(y'')\big)\big(x'_2 x''_2, s(y')\big) \\
&= \sum c^{|x'_2| \cdot |x''_1|} \big(x'_1 x''_1 \otimes x'_2 x''_2, c^{|y'| \cdot |y''|} s(y'') \otimes s(y')\big) \\
&= \big(\delta(x), c^{|y'| \cdot |y''|} s(y'') \otimes s(y')\big) \\
&= (x, s(y'y'')).
\end{aligned}$$

(iii) Let $L' = (L', m', e', \delta', \varepsilon', s')$ with generators u'_i, $i \in I$. We need to prove $\varepsilon'\psi = \varepsilon$, $\delta'\psi = (\psi \otimes \psi)\delta$, and $\psi s = s'\psi$.

Since ψ is a graded map, it follows that $\varepsilon'\psi = \varepsilon$.

Since both $\delta'\psi$ and $(\psi \otimes \psi)\delta$ are K-algebra homomorphisms from L to $(L' \otimes L')_\chi$, it suffices to check their values on the generators u_i, $i \in I$, which are clearly same.

It remains to check $\psi s(u_a) = s'\psi(u_a)$ for all $a \in I(\mu)$ and all μ. Since $\psi(u_a) = u'_a$, the assertion follows from Lemma 4.3.

(iv) By the argument in [L, 1.2.6] we see that $L/\mathscr{I}$ also becomes a $(K, c, I, \cdot)$-bialgebra satisfying **L1** and **L3**. Since s is invertible by Theorem 2.10(v), it follows from (ii) that $s(\mathscr{I}) = \mathscr{I}$, and hence s is also the antipode of $L/\mathscr{I}$.

(v) Just repeat the proof of Theorem 3.7(i). ∎

4.5. Example. As in Example 2.12, we know that Lusztig's algebra $'\mathbf{F} = K\langle I \rangle$ is a free, $(K, c, I, \cdot)$-Hopf algebra, say, with antipode s. On the other hand, by the argument in [L, Proposition 1.2.3] we see that $'\mathbf{F}$ also satisfies **L3**. Then by Theorem 4.4(iv) $\mathbf{F} = {}'\mathbf{F}/\mathscr{I} \in \mathscr{L}(K, c, I, \cdot)$ with the antipode induced by s.

The following lemma due to Green gives a common property of the algebras in Green's class with datum $(I, \cdot)$, see [G1, Proposition 3.2(a); G2, Proposition 2.3].

Lemma 4.6 (Green). *Let $(I, \cdot)$ be a datum, let $x = (x_i)_{i \in I} \in \mathbb{N}_0 I$, and let $a, b \in I(x)$. Then there exists a Laurent polynomial $M_{a,b}(t) \in \mathbb{Z}[t, t^{-1}]$ with t an indeterminate such that for any K, c as in 4.1 and for any $L \in \mathscr{L}(K, c, I, \cdot)$ with generators u_i, $i \in I$, there holds*

$$(u_a, u_b) = M_{a,b}(c) B_x(L) \tag{4.6}$$

with

$$B_x(L) = \prod_{i \in I} (u_i, u_i)^{x_i}.$$

The polynomials $M_{a,b}(t)$ will be called the Green polynomials. One of the advantages of the Green polynomials is that they depend only on the datum $(I, \cdot)$, but not on K, c, nor on L; and also the bilinear form $(-, -)$ on $L \in \mathscr{L}(K, c, I, \cdot)$ depends only on the values (u_i, u_i), $i \in I$, not on K, nor on c. Of course, $M_{a,b}(t)$ is symmetric, i.e., $M_{a,b}(t) = M_{b,a}(t)$.

The following result implies that Green's polynomials enjoy some cyclic-symmetry.

Theorem 4.7. *Let $(I, \cdot)$ be an arbitrary datum. Let $x = (x_i)_{i \in I} \in \mathbb{N}_0 I$, and $a, b \in I(x)$. Then*

$$M_{\sigma(a), b}(t) = M_{a, \sigma(b)}(t). \tag{4.7}$$

Proof. Choose K to the field of real numbers. Consider the corresponding Lusztig algebra $\mathbf{F}$ in $\mathscr{L}(K, c, I, \cdot)$. Since both $a, b \in I(x)$, it follows from Lemma 4.3 that $s(\theta_a) = \lambda(x)\theta_{\sigma(a)}$ and $s(\theta_b) = \lambda(x)\theta_{\sigma(b)}$, and hence by Theorem 4.4(ii) we have

$$\lambda(x)\big(\theta_{\sigma(a)}, \theta_b\big) = \big(s(\theta_a), \theta_b\big) = \big(\theta_a, s(\theta_b)\big) = \lambda(x)\big(\theta_a, \theta_{\sigma(b)}\big),$$

and hence $(\theta_{\sigma(a)}, \theta_b) = (\theta_a, \theta_{\sigma(b)})$. It follows from Lemma 4.6 that

$$M_{\sigma(a), b}(c) B_x(\mathbf{F}) = \big(\theta_{\sigma(a)}, \theta_b\big) = \big(\theta_a, \theta_{\sigma(b)}\big) = M_{a, \sigma(b)}(c) B_x(\mathbf{F});$$

thus $M_{\sigma(a), b}(c) = M_{a, \sigma(b)}(c)$ for all non-zero real numbers c. It follows that (4.7) holds. ∎

By [G2, Proposition 2.5] and Theorem 4.4(iii) we have

LEMMA 4.8. *Any two non-degenerate members L and L' in $\mathscr{L}(K, c, I, \cdot)$ are isomorphic as $(K, c, I, \cdot)$-Hopf algebras by sending the generators u_i of L to the generators u'_i of L', $i \in I$.*

Assume that $(I, \cdot)$ is a Cartan datum in Lusztig's sense; i.e., I is a finite set, and $\cdot$ is a symmetric bilinear form on $\mathbb{Z}I$ such that $\frac{i \cdot i}{2}$ is a positive integer for any $i \in I$, and that $2\frac{i \cdot j}{i \cdot i}$ is a non-positive integer for any $i \neq j$ in I.

Notice that for any finite field with $|k| = q_k$, there exists a finite dimensional hereditary k-algebra Λ of type $(I, \cdot)$; i.e., the symmetrization $(-,-)$ of the Ringel form of Λ is exactly $\cdot$, namely, $(S(i), S(j)) = i \cdot j$ for $i \in I$. Set $v_k = \sqrt{q_k}$. Then $\mathscr{C}(\Lambda) \in \mathscr{L}(\mathbb{Q}(v_k), v_k, I, \cdot)$.

Consider the Lusztig algebra $\mathbf{F}$ associated with $(I, \cdot)$ over $\mathbb{Q}(v_k)$. The following isomorphism π can be regarded as the non-generic case of the Ringel–Green isomorphism described in [G1, Theorem 3].

COROLLARY 4.9. *There is a unique $(\mathbb{Q}(v_k), v_k, I, \cdot)$-Hopf algebra isomorphism*

$$\pi : \mathscr{C}(\Lambda) \cong \mathbf{F} \tag{4.8}$$

with $\pi([S(i)]) = \theta_i$, $i \in I$. Denote the bilinear forms on $\mathscr{C}(\Lambda)$ and on $\mathbf{F}$ by $(-,-)_1$ and $(-,-)_2$, respectively. Then we have

$$(\pi(a), \pi(b))_2 = (a, b)_1, \qquad \forall a, b \in \mathscr{C}(\Lambda). \tag{4.9}$$

Proof. Since both $\mathscr{C}(\Lambda)$ and $\mathbf{F}$ are non-degenerate objects in $\mathscr{L}(\mathbb{Q}(v_k), v_k, I, \cdot)$, it follows from Lemma 4.8 that there is a unique $(\mathbb{Q}(v_k), v_k, I, \cdot)$-Hopf algebra isomorphism π: $\mathscr{C}(\Lambda) \cong \mathbf{F}$ with $\pi([S(i)]) = \theta_i$, $i \in I$. In order to see the formula (4.9), by Lemma 4.6 one only needs to observe the fact that

$$([S(i)],[S(i)])_1 = \frac{v_k^{i\cdot i}}{v_k^{i\cdot i}-1} = (\theta_i,\theta_i)_2, \qquad i\in I.$$

∎

4.10. Let $\mathbb{Q}(v)$ be the functional field with indeterminate v. Let $(I,\cdot)$ be a Cartan datum, and let U be the Drinfeld–Jimbo quantized enveloping algebra of type $(I,\cdot)$ with positive part U^+. By definition U^+ is the $\mathbb{Q}(v)$-algebra with generators E_i, $i\in I$, and with quantum Serre relations as defining relations. It is proved by Lusztig [L, Theorem 33.1.3] that there is a unique $\mathbb{Q}(v)$-algebra isomorphism $U^+\cong \mathbf{f}$ sending E_i to θ_i, $i\in I$, where $\mathbf{f}$ is the Lusztig algebra associated with $(I,\cdot)$ over $\mathbb{Q}(v)$. This means that $\mathbf{f}={}'\mathbf{f}/\mathscr{I}$, where $\mathscr{I}$ is exactly the two-sided ideal generated by the elements θ_{ij}^+ for $i\neq j$ in I,

$$\theta_{ij}^+ = \sum_{0\le t\le 1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{i\cdot i/2} \theta_i^{1-a_{ij}-t}\theta_j\theta_i^t, \qquad i\neq j\in I, \quad (4.10)$$

where $a_{ij}=2\frac{i\cdot j}{i\cdot i}$.

Let $\mathscr{C}(I,\cdot)$ be Ringel's twisted generic composition algebra of type $(I,\cdot)$; see [R2, R4] for details. The famous Ringel–Green isomorphism

$$\mathscr{C}(I,\cdot)\cong \mathbf{f}\cong U^+$$

permits us to define $(-,-)$ and s on $\mathscr{C}(I,\cdot)$ and U^+ via $\mathbf{f}$.

Summarize the discussions above as follows

THEOREM 4.11 (Ringel–Green). *Let $(I,\cdot)$ be a Cartain datum. Then*

(1) *$U^+,\mathbf{f},\mathscr{C}(I,\cdot)\in\mathscr{L}(\mathbb{Q}(v),v,I,\cdot)$, and there is a unique $(\mathbb{Q}(v),v,\cdot)$-Hopf algebra isomorphism*

$$U^+\cong \mathbf{f}\cong \mathscr{C}(I,\cdot)$$

sending generators to generators.

(2) *Let H denote any one of U^+, $\mathbf{f}$, and $\mathscr{C}(I,\cdot)$, and let s denote the corresponding antipode. Then $s\colon H\to H_\chi$ gives an $\mathbb{N}_0 I$-graded, $\mathbb{Q}(v)$-algebra anti-isomorphism, where $\chi=\cdot$, and the multiplication m_χ of H_χ is given by Ringel's rule (2.1) in 2.3.*

5. EXAMPLES

For the general representation theory of hereditary algebras we refer the reader to [ARS, R1].

Let k be a finite field with $|k| = q$. To be simple, set $v = \sqrt{q}$. For the calculations below, we need the following facts (see, e.g., [Z1, Z2]).

LEMMA 5.1. *Let Λ be a finite dimensional hereditary k-algebra. Then*

(1) *If $L = M \oplus N$ with M and N having no direct summands in common, then*

$$a_L = a_M a_N q^{\dim_k \mathrm{Hom}(M,N) + \dim_k \mathrm{Hom}(N,M)}.$$

(2) *Assume that* $\mathrm{Hom}_k(N, M) = 0$. *If $g^L_{M,N} \neq 0$, then $g^L_{M,N} = 1$.*

(3) *We have*

$$g^{M\oplus N}_{M,N} = \frac{a_{M\oplus N}}{a_M a_N q^{\dim_k \mathrm{Hom}_A(M,N)}}.$$

In particular, if M and N have no direct summands in common, then

$$g^{M\oplus N}_{M,N} = q^{\dim_k \mathrm{Hom}_\Lambda(N, M_1)}.$$

5.2. Let Q be the quiver with two vertices denoted by 1 and 2, and n arrows from 1 to 2, and let $\Lambda = kQ$, the path algebra of Q. Then $\langle S(1), S(2)\rangle = 0$, $\langle S(2), S(1)\rangle = -n$. We have $(q^n - 1)/(q-1)$ indecomposable modules with dimension vector $(1,1)$, say, $N_1, \ldots, N_{(q^n-1)/(q-1)}$. Then $a_{N_i} = q - 1$. Set $r_1 = \Sigma_{1\le i\le (q^n-1)/(q-1)}[N_i]$. Then we have

$$[S(1)][S(2)] = [S(1)\oplus S(2)],$$
$$[S(2)][S(1)] = v^{-n}[S(1)\oplus S(2)] + v^{-n} r_1.$$

By the formula (3.4) in Section 3 we have

$$\begin{aligned}
\delta([S(1)\oplus S(2)]) &= [S(1)\oplus S(2)]\otimes 1 + 1 \oplus [S(1)\oplus S(2)] \\
&\quad + [S(1)]\otimes[S(2)] + v^{-n}[S(2)]\otimes[S(1)],\\
\delta([N_i]) &= [N_i]\otimes 1 + 1\otimes [N_i] + v^{-n}(q-1)[S(2)]\otimes[S(1)],\\
\delta(r_1) &= r_1\otimes 1 + 1\otimes r_1 + v^{-n}(q^n-1)[S(2)]\otimes[S(1)].
\end{aligned}$$

By Lemma 3.3 we have

$$\begin{aligned} s([S(1) \oplus S(2)]) &= -[S(1) \oplus S(2)] \\ &\qquad + (v^{-2n} + 1)[S(1) \oplus S(2)] + v^{-2n} r_1 \\ &= v^{-2n}[S(1) \oplus S(2)] + v^{-2n} r_1, \end{aligned}$$

$$s([N_i]) = -[N_i] + v^{-2n}(q-1)[S(1) \oplus S(2)] + v^{-2n}(q-1) r_1.$$

Thus,

$$s(r_1) = -v^{-2n} r_1 + v^{-2n}(q^n - 1)[S(1) \oplus S(2)].$$

This verifies

$$\begin{aligned} s([S(1)][S(2)]) &= s([S(1) \oplus S(2)]) \\ &= v^{-2n}[S(1) \oplus S(2)] + v^{-2n} r_1 \\ &= [S(2)] * [S(1)], \end{aligned}$$

$$\begin{aligned} s([S(2)][S(1)]) &= v^{-n}\big(s([S(1) \oplus S(2)]) + s(r_1)\big) = v^{-n}[S(1) \oplus S(2)] \\ &= v^{-n}[S(1)][S(2)] = [S(1)] * [S(2)], \end{aligned}$$

and

$$\big(s([S(1) \oplus S(2)]), [N_i]\big) = \big([S(1) \oplus S(2)], s([N_i])\big) = \frac{q^{2-n}}{q-1}.$$

It follows that

$$\big(s([S(1) \oplus S(2)]), r_1\big) = \big([S(1) \oplus S(2)], s(r_1)\big) = \frac{q^n - 1}{(q-1)^2 q^{n-2}}.$$

Also, $[N_i] - [N_{i-1}]$, $2 \leq i \leq (q^n - 1)/(q-1)$, is a basis of the subspace of $\mathscr{H}(\Lambda)_{(1,1)}$ consisting of δ-primitive elements.

5.3. Let Λ be a finite dimensional hereditary algebra of tame type, and let E be a quasi-simple (regular) module. Then we have

$$s([E]) + [E] \in \mathscr{C}(\Lambda).$$

In fact, let $0 \to M \to E \to N \to 0$ be an exact sequence with $M \neq 0 \neq N$. Since E is quasi-simple, it follows that M must be preprojective and N must be preinjective, and hence

$$\delta([E]) = 1 \otimes [E] + [E] \otimes 1 + \sum c_{P,I}([I] \otimes [P]),$$

where P are preprojective, I is preinjective, and $c_{P,I} \in \mathbb{Q}(v)$. Thus, by $m(\mathrm{id} \otimes s)\delta = e\varepsilon$, we have $s([E]) + [E] = -\sum c_{P,I}([I]s([P])) \in \mathscr{C}(\Lambda)$, where we have used the fact that $[I], s([P[) \in \mathscr{C}(\Lambda)$; see [Z2, Z3].

APPENDIX

Let $A = (A, m, e, \delta, \varepsilon)$ be a (K, c, I, χ)-bialgebra, with χ not necessarily symmetric. Define for each $x \in \mathbb{Z}I$ a symbol K_x, and let T be the K-vector space with basis $\{K_x \mid x \in \mathbb{Z}I\}$. Make T into a K-algebra by defining

$$K_x K_y = K_{x+y}; \tag{A1}$$

i.e., T is the group algebra over K of the additive group $\mathbb{Z}I$.

Let TA be the K-algebra generated by the algebra A and the symbols K_x, with defining relations (A1) and

$$aK_x = c^{-\chi(x,|a|)} K_x a \tag{A2}$$

for all $x \in \mathbb{Z}I$, $a \in A$ (a homogeneous) (cf. [L, p. 19, 3.1.1(b); X, p. 122, 4.5(c)]. Then TA is $\mathbb{N}_0 I$-graded with $(TA)_x = TA_x$.

Next we define a comultiplication Δ: $TA \to TA \otimes TA$ in such a way that Δ is a K-algebra map, where $TA \otimes TA$ is the K-algebra by the usual rule, i.e., $(u_1 \otimes u_2)(v_1 \otimes v_2) = u_1 v_1 \otimes u_2 v_2$, for all $u_1, u_2, v_1, v_2 \in TA$. For this purpose we set

$$\Delta(K_x) = K_x \otimes K_x, \qquad \text{for all } x \in \mathbb{Z}I, \tag{A3}$$

and

$$\Delta(a) = \sum a_1 K_{|a_2|} \otimes a_2 \tag{A4}$$

for all $a \in A$, with $\delta(a) = \sum a_1 \otimes a_2$ (a_1, a_2 homogeneous). To prove that these definitions are well defined, we must show that they respect the relations which define TA. Thus we must check that

$$\Delta(K_x)\Delta(K_y) = \Delta(K_{x+y}) \qquad \text{for all } x, y \in \mathbb{Z}I, \tag{A5}$$

$$\Delta(a)\Delta(b) = \Delta(ab) \qquad \text{for all } a, b \in A, \tag{A6}$$

and

$$\Delta(a)\Delta(K_x) = c^{-\chi(x,|a|)}\Delta(K_x)\Delta(a), \tag{A7}$$

for all $x \in \mathbb{Z}I$ and all homogeneous $a \in A$.

Equations (A5) and (A7) are straightforward. To check (A6), we remember that

$$\delta(ab) = \delta(a) * \delta(b) = \sum c^{(|a_2|,|b_1|)} a_1 b_1 \otimes a_2 b_2 .$$

Therefore by (A4) and (A2) we have

$$\begin{aligned} \Delta(ab) &= \sum c^{(|a_2|,|b_1|)} a_1 b_1 K_{|a_2|} K_{|b_2|} \otimes a_2 b_2 \\ &= \sum a_1 K_{|a_2|} b_1 K_{|b_2|} \otimes a_2 b_2 \\ &= \Big(\sum a_1 K_{|a_2|} \otimes a_2 \Big) \Big(\sum b_1 K_{|b_2|} \otimes b_2 \Big) \\ &= \Delta(a) \Delta(b). \end{aligned}$$

To prove Δ is coassociative, it is enough to show that

$$(\mathrm{id} \otimes \Delta)\Delta(u) = (\Delta \otimes \mathrm{id})\Delta(u) \tag{A8}$$

holds for all $u = K_x$ and all $u = a \in A$. The verifications are straightforward.

Finally we extend the counit ε: $A \to K$ to a K-algebra map ε: $TA \to K$ by setting $\varepsilon(K_x a) = \varepsilon(a)$ for all $x \in \mathbb{Z}I$, $a \in A$. One verifies that ε, so defined, is multiplicative (use (A1) and (A2)), and that it is a counit for Δ. We have now proved part (i) of the following

THEOREM A. *Let $A = (A, m, e, \delta, \varepsilon)$ be a (K, c, I, χ)-bialgebra, χ not necessarily symmetric. Define TA as an $\mathbb{N}_0 I$-graded space, with multiplication, comultiplication, and counit as above. Then*

(i) *TA is a K-bialgebra in the usual sense, and*

(ii) *TA is a K-Hopf algebra in the usual sense, with antipode s': $TA \to TA$ given by*

$$s'(K_x) = K_{-x}, \qquad \textit{for all } x \in \mathbb{Z}I, \tag{A9}$$

and

$$s'(a) = K_{-|a|} s(a), \qquad \textit{for all homogeneous } a \in A, \tag{A10}$$

where s is the antipode of A.

Remark. (a) Notice that TA is an $\mathbb{N}_0 I$-graded K-Hopf algebra, but with $(TA)_0 = T$; therefore TA is not a $(K, c, I, 0)$-Hopf algebra (this would require $(TA)_0 = K$).

(b) Since the antipode s of A is invertible, s' is also invertible with $s'^{-1}(K_x) = K_{-x}$ and $s'^{-1}(a) = K_{|a|} s^{-1}(a)$.

Proof of Theorem A, *part* (ii). We first show that (A9) and (A10) can be uniquely extended to an anti-homomorphism $s'\colon TA \to TA$. For this, we must verify

$$s'(K_x)s'(K_y) = s'(K_{x+y}) \qquad \text{for all } x, y \in \mathbb{Z}I, \tag{A11}$$

$$s'(a)s'(b) = s'(ba) \qquad \text{for all } a, b \in A, \tag{A12}$$

and

$$s'(K_x)s'(a) = c^{-\chi(x,|a|)}s'(a)s'(K_x) \tag{A13}$$

for all $x \in \mathbb{Z}I$ and all homogeneous $a \in A$. Both (A11) and (A13) can be easily checked. For (A12), we use Theorem 2.10(iii), which tells us that $s(ba) = s(a) * s(b)$, where $*$ is the product in A_{χ^T}. So $s(ba) = c^{\chi(|b|,|a|)}s(a)s(b)$. Therefore,

$$\begin{aligned} s'(a)s'(b) &= K_{-|a|}s(a)K_{-|b|}s(b) = c^{-\chi(-|b|,|a|)}K_{-|a|-|b|}s(a)s(b) \\ &= K_{-|ba|}s(ba) = s'(ba). \end{aligned}$$

It remains to show that $M(\mathrm{id} \otimes s')\Delta(u) = e\varepsilon(u) = M(s' \otimes \mathrm{id})\Delta(u)$, for all $u = K_x$ $(x \in \mathbb{Z}I)$ and all $u = a \in A$ (a homogeneous), where M denotes the multiplication map of TA. For $u = K_x$ this is trivial. If $u = a$ we have

$$\begin{aligned} M(\mathrm{id} \otimes s')\Delta(a) &= \sum a_1 K_{|a_2|}s'(a_2) = \sum a_1 K_{|a_2|}K_{-|a_2|}s(a_2) = \sum a_1 s(a_2) \\ &= \varepsilon(a). \end{aligned}$$

On the other hand we get

$$\begin{aligned} M(s' \otimes \mathrm{id})\Delta(a) &= \sum s'(a_1 K_{|a_2|})a_2 = \sum s'(K_{|a_2|})s(a_1)a_2 \\ &= \sum K_{-|a_2|}K_{-|a_1|}s(a_1)a_2 = K_{-|a|}\sum s(a_1)a_2 = K_{-|a|}\varepsilon(a). \end{aligned}$$

If $a \in A_x$, $x \neq 0$, this is zero, which equals $\varepsilon(a)$. If $a \in A_0$, it is $K_0\varepsilon(a) = \varepsilon(a)$. ∎

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REFERENCES

[ARS] M. Auslander, I. Reiten, and S. Smalø, "Representation Theory of Artin Algebras," Cambridge Studies in Advanced Math., Vol. 36, Cambridge Univ. Press, Cambridge, UK, 1994.

[D] V. G. Drinfeld, Hopf algebras and quantum Yang–Baxter equation, *Soviet. Math. Dokl.* **32** (1985), 254–258.

[G1] J. A. Green, Hall algebras, hereditary algebras and quantum groups, *Invent. Math.* **120** (1995), 361–377.

[G2] J. A. Green, Quantum groups, Hall algebras, and quantized shuffles, *in* "Finite Reductive Groups, Related Structures and Representations" (M. Cabanes, Ed.), Birkhäuser, Boston, 1996.

[Jan] J. C. Jantzen, "Lectures on Quantum Groups," Graduate Studies in Math., Vol. 6, Am. Math. Soc., Providence, 1995.

[Jim] M. Jimbo, A q-difference analogue of $U(g)$ and the Yang–Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69.

[Jos] A. Joseph, "Quantum Groups and Their Primitive Ideals," Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 29, Springer-Verlag, Berlin/New York, 1995.

[K] V. G. Kac, "Infinite Dimensional Lie Algebras," Progress on Math., Vol. 44, Birkhäuser, Boston/Basel/Stuttgart, 1983.

[L] G. Lusztig, "Introduction to Quantum Groups," Progress on Math., Vol. 110, Birkhäuser, Boston/Basel/Berlin, 1993.

[M] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," 2nd ed., Oxford Univ. Press, London, 1995.

[R1] C. M. Ringel, "Tame Algebras and Integral Quadratic Forms," Lecture Notes in Math., Vol. 1099, Springer-Verlag, New York/Berlin 1984.

[R2] C. M. Ringel, Hall algebras, *Banach Center Publ.* **26** (1990), 433–447.

[R3] C. M. Ringel, Hall algebras and quantum groups, *Invent. Math.* **101** (1990), 583–592.

[R4] C. M. Ringel, Hall algebras revisited, *Israel Math. Conf. Proc.* **7** (1993), 171–176.

[R5] C. M. Ringel, PBW-bases of quantum groups, *J. Reine Angew. Math.* **470** (1996), 51–88.

[R6] C. M. Ringel, Green's theorem on Hall algebras, *Canad. Math. Soc. Conf. Proc.* **19** (1996), 185–245.

[Ro] M. Rosso, Quantum groups and quantum shuffles, *Invent. Math.* **133** (1998), 399–416.

[SV] B. Sevenhant and M. Van den Bergh, A relation between a conjecture of Kac and the structure of the Hall algebra, preprint.

[S] M. Sweedler, "Hopf Algebra," Benjamin, New York, 1969.

[X] J. Xiao, Drineld double and Ringel–Green theory, *J. Algebra* **190** (1997), 100–144.

[Z] A. Zelevinski, "Representations of Finite Classical Groups," Lecture Notes in Math., Vol. 869, Springer-Verlag, New York/Berlin, 1981.

[Z1] P. Zhang, Triangular decomposition of the composition algebra of the Kronecker algebra, *J. Algebra* **184** (1996), 159–174.

[Z2] P. Zhang, Composition algebras of affine type, *J. Algebra* **206** (1998), 505–540.

[Z3] P. Zhang, Representations as elements in affine composition algebras, *Trans. Amer. Math. Soc.*, to appear.